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Nonclassical dynamical thermoelasticity

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Abstract

This paper describes the modern approaches to the analytical treatment of dynamical thermoelasticity. It has been a well known fact that the classical heat conduction equation does not describe the phenomenon of heat propagation correctly. Contrary to the solutions of the classical heat conduction equation, which predicts the infinite speed of the heat wave, the experimental results indicate that heat travels with finite speed. The same results apply to thermoelastic waves. Therefore, the modern approaches to this problem depend on appropriate modifications of the classical heat conduction equation. Five such approaches are described in the paper: (a) Lord and Shulman (L–S) theory; (b) Green and Lindsay (G–L) theory; (c) Hetnarski and Ignaczak (H–I) theory; (d) Green and Naghdi (G–N) theory; and (e) Chandrasekharaiah and Tzou (C–T) theory. Some evaluation and comparison of the results that follow from these five descriptions is provided.

1. Introduction

By a nonclassical dynamical thermoelasticity we mean a hyperbolic thermoelasticity in which disturbances propagate with finite wave speeds. The first theories of hyperbolic thermoelasticity were formulated in the late 1960's in an attempt to eliminate the shortcomings of classical thermoelasticity, such as:

1. infinite velocity of thermoelastic disturbances,
2. unsatisfactory thermoelastic response of a solid to short laser pulses, and
3. poor description of thermoelastic behavior at low temperatures (Lord and Shulman, 1967; Green and Lindsay, 1972; Francis, 1972; Chandrasekharaiah, 1986).

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Attempts to present a theory of thermoelastic waves that would be attractive to both the basic and applied researchers have been continued in the literature up to date.

The aim of this paper is to review a number of results on finite wave speed thermoelastic disturbances that are based on five different theories of a thermoelastic solid, and which have been obtained in the literature during the past 30–33 years. The five theories discussed are:

(i) Generalized Thermoelasticity proposed in 1967 by Lord and Shulman (L–S Theory),
(ii) Temperature-Rate Dependent Thermoelasticity introduced in 1972 by Green and Lindsay (G–L Theory),
(iii) Low-Temperature Thermoelasticity proposed in 1996 by Hetnarski and Ignaczak (H–I Theory),
(iv) Thermoelasticity Without Energy Dissipation formulated in 1993 by Green and Naghdi (G–N Theory), and
(v) Dual-Phase-Lag Thermoelasticity proposed in 1998 by Chandrasekharaiah and Tzou (C–T Theory).

Although the five theories are not the only ones that have been proposed during the past 30–33 years in an attempt to describe thermoelastic waves in a solid, they are, in the authors' opinion, representative in discussing the subject. In particular, in each of the five theories, a thermoelastic disturbance produced by an external thermomechanical load of a bounded support cannot invade an unbounded body in finite time.

It should be stressed that these theories have not been verified by an experiment up to date. To study concrete thermoelastic waves described by a solution to a particular initial-boundary value problem in any of these theories, approximate values of the material functions are usually used. For example, to numerically discuss an analytical solution to a problem in L–S theory, the value of a relaxation time is taken from experimental results for a rigid heat conductor, while the remaining material functions are identified with those of an elastic body under isothermal or adiabatic conditions. Finding materials that would comply with any of the five hyperbolic thermoelastic theories remains a challenge for experimental researchers in the field of thermoelastic waves.

2. A thermoelastic wave propagating in the L–S model

The L–S model is described by a system of partial differential equations (PDE) in which in comparison to a system of classical thermoelasticity, the Fourier law of heat conduction is replaced by the Maxwell–Cattaneo law that generalizes the Fourier law and introduces a single relaxation time into consideration. An ordered array of functions $[\mathbf{u}, \mathbf{E}, \mathbf{S}; \theta, \eta, \mathbf{q}]$; in which $\mathbf{u}$, $\mathbf{E}$, $\mathbf{S}$, θ, η and $\mathbf{q}$ denote the displacement, strain, stress, temperature, entropy and heat flux fields, respectively; that comply with system of PDE describes a thermoelastic wave propagating in the L–S model. These fields are defined on a cartesian product $\bar{B} \times [0,\infty)$, where $\bar{B}$ is a domain occupied by the model and $[0, \infty)$ is the time interval. By eliminating four functions from the six that define a wave, one can obtain the field equations of L–S theory in terms of various pairs of mechanical and thermal variables, such as $(\mathbf{u}, \theta)$, $(\mathbf{u}, \mathbf{q})$, $(\mathbf{S}, \theta)$ and $(\mathbf{S}, \mathbf{q})$; and a pair of thermomechanical variables (,), formed from the variables that define a thermoelastic wave, corresponds to the wave if the remaining variables of the wave can be restored from the pair. For example, a pair $(\mathbf{u}, \theta)$ that satisfies the displacement-temperature field equations of L–S theory, subject to suitable initial and boundary conditions, is a pair corresponding to a thermoelastic wave because it generates the fields $\mathbf{E}$, $\mathbf{S}$, η and $\mathbf{q}$ in such a way that the ordered array of functions $[\mathbf{u}, \mathbf{E}, \mathbf{S}; \theta, \eta, \mathbf{q}]$ represents a thermoelastic wave corresponding to an external thermomechanical load applied to the body $\bar{B}$ over a time interval. An initial-boundary value problem for a pair ($\mathbf{u}$, θ) in which the initial conditions are imposed on the displacement $\mathbf{u}$, velocity $\dot{\mathbf{u}}$, temperature θ and temperature rate $\dot{\theta}$, is called a displacement-temperature characterization of a thermoelastic wave of L–S theory.

Similarly, a pair ($\mathbf{S}$, $\mathbf{q}$) that satisfies the stress-heat flux field equations of L–S theory subject to suitable stress-heat flux initial and boundary conditions is a pair corresponding to a thermoelastic wave because it generates the fields $\mathbf{u}$, $\mathbf{E}$, θ and η in such a way that the array of functions [$\mathbf{u}$, $\mathbf{E}$, $\mathbf{S}$; θ, η, $\mathbf{q}$] represents a thermoelastic wave of L–S theory (Ignaczak, 1989).

For a nonhomogeneous anisotropic thermoelastic body, a displacement-temperature wave corresponding to an external thermomechanical load depends on the following set of constitutive variables:

$$\{\theta_0,\, t_0;\, \rho,\, c_E;\, \mathbf{K},\, \mathbf{M};\, \mathbf{C}\}. \tag{1}$$

Here, θ_0 and t_0 are a fixed uniform reference temperature and a constant relaxation time, respectively; $\rho=\rho(x)$ and $c_E=c_E(x)$ are the mass density and the specific heat for zero strain (scalar) fields, respectively; $\mathbf{K}=\mathbf{K}(x)$ and $\mathbf{M}=\mathbf{M}(x)$ are the conductivity and the stress-temperature (second-order) tensor fields, respectively; and $\mathbf{C}=\mathbf{C}(x)$ is the elasticity (fourth order) tensor field.

The external thermomechanical load in a mixed displacement-temperature problem is represented by the set of functions:

$$\{\mathbf{b},\, r;\, \mathbf{u}_0,\, \dot{\mathbf{u}}_0,\, \vartheta_0,\, \dot{\vartheta}_0;\, \mathbf{u}',\, \mathbf{s}',\, \theta',\, q'\}. \tag{2}$$

Here, $\mathbf{b}=\mathbf{b}(x,t)$ and $r=r(x,t)$ are the body force and heat supply fields; respectively; ($\mathbf{u}_0$, ϑ_0) and ($\dot{\mathbf{u}}_0$, $\dot{\vartheta}_0$) are the initial values of ($\mathbf{u}$, θ) and ($\dot{\mathbf{u}}$,$\dot{\theta}$), respectively; and $\mathbf{u}'$, $\mathbf{s}'$ θ' and q' denote the boundary displacement, traction, temperature, and heat flux, respectively. Moreover, the superimposed dot denotes the partial derivative with respect to time t.

Let $B(t)$ denote a support of the thermomechanical load Eq. (2) for a fixed time t, i.e. the set of points of $\bar{B}$ on which the load does not vanish over the interval [0, t]. Let $\mathbf{A}=\mathbf{A}(x,\, \mathbf{m})$ be the (second order) acoustic tensor in the propagation direction $\mathbf{m}$ defined for any unit vector $\mathbf{m}$ and any vector $\mathbf{a}$ by the relation

$$\mathbf{A}(x,\, \mathbf{m})\mathbf{a} = \rho^{-1}(x)\mathbf{C}[\mathbf{a}\otimes\mathbf{m}]\mathbf{m}, \tag{3}$$

where $\rho=\rho(x)$ and $\mathbf{C}=\mathbf{C}(x)$ are the mass density and elasticity tensor fields, respectively, (cf. Eq. (1)). Moreover, let $S(x,\, Ct)$ denote an open ball in E^3 with radius Ct and center at x. Finally, let $B^*(t)$ be the set defined by

$$B^*(t) = \{x\in\bar{B}\colon B(t)\cap\overline{S(x,\, Ct)} \neq \varnothing\}. \tag{4}$$

The following theorem shows that the displacement-temperature wave produced by the external load Eq. (2) propagates with a finite speed.

Theorem 1 (Domain of influence theorem for mixed displacement-temperature problem of L–S theory)**.** Let ($\mathbf{u}$, θ) be a solution to the mixed problem. Then for $C\geq\max(C_1,\, C_2)$,

$$\mathbf{u}=0,\, \theta=0 \text{ on } [\bar{B}-B^*(t)]\times[0,\, t]. \tag{5}$$

Here, C_1 is an upper bound over B and $|\mathbf{m}|=1$ of a simple algebraic function that depends on θ_0, ρ_0, c_E, $|\mathbf{M}|$ and $|\mathbf{A}|$; while C_2 is an upper bound over B of a simple algebraic function that depends on θ_0, t_0, ρ, c_E, $|\mathbf{M}|$ and $|\mathbf{K}|$.

Proof (Given by Ignaczak et al., 1986 and Ignaczak, 1991)**.** The set $B^*(t)$ is called a domain of influence of the load Eq. (2) for the mixed problem.

Theorem 1 implies that for a finite t and for a bounded support of the thermomechanical load, i.e. for a bounded set $B(t)$, the thermoelastic disturbance generated by a pair $(\mathbf{u}, \theta)$ vanishes outside of a bounded domain $B^*(t)$ that depends on the load support, the bounds on the thermomechanical constitutive fields (see Eq. (1)), and the relaxation time t_0. This theorem also shows that the thermoelastic disturbance propagates as a wave from the domain $B(t)$ with a finite speed equal to or less than the speed C. An analysis of the velocities C_1 and C_2 indicates that C_1 is finite and $C_2 \to \infty$ as $t_0 \to 0+0$; hence, $C \to \infty$ as $t_0 \to 0+0$. Therefore, if the relaxation time goes to zero, the thermoelastic disturbance described by the pair $(\mathbf{u}, \theta)$ gains an infinite speed, as should be expected since, in this limiting case, the mixed displacement-temperature problem of L–S theory reduces to a mixed problem of classical thermoelasticity.

An analysis of the velocities C_1 and C_2 also shows that for a particular nonhomogeneous anisotropic L–S model in which the acoustic and conductivity tensor fields are relatively small, the maximum speed of a thermoelastic wave is given by the formula

$$C_0 = \sup_B \left\{ \left(\frac{\theta_0}{\rho c_E} \right)^{\frac{1}{2}} |\mathbf{M}| \right\}. \tag{6}$$

This formula shows that for a nonhomogeneous anisotropic thermoelastic body in which the acoustic tensor and the heat conductivity tensor fields are relatively small, the maximum speed of a thermoelastic wave in the L–S theory is dominated by a suitably scaled stress-temperature tensor field.

Also, the analysis of C_1 and C_2 shows that if $|\mathbf{M}|$ is relatively small, the velocity C_1 reduces to that of a domain of influence theorem from classical isothermal elastodynamics (Gurtin, 1972; Eringen and Suhubi, 1975), while the velocity C_2 reduces to that of a domain of influence theorem for a nonhomogeneous anisotropic rigid heat conductor.

Finally, for a finite value of $|\mathbf{M}|$, C_1 and C_2 represent upper bounds on the velocities of a quasimechanical and of a quasithermal wave, respectively, propagating in the L–S model.

As far as the stress-heat flux characterization of a wave in the L–S model is concerned, we note that a domain of influence result for a pair $(\mathbf{S}, \mathbf{q})$ was formulated and proved by Biały (1991); the result covers the case of a thermoelastic wave produced by the initially distributed thermoelastic defects in an L–S model that is not included in Theorem 1.

3. A thermoelastic wave propagating in the G–L model

The G–L model is characterized by a system of partial differential equations in which, in comparison to the classical system, the constitutive relations for the stress tensor and the entropy are generalized by introducing two different relaxation times into considerations.

A displacement-temperature wave propagating in the G–L model complies with the system of field equations for a pair $(\mathbf{u}, \theta)$ subject to the initial and boundary conditions similar to that of L–S theory. The existence of two relaxation times, t_0 and t_1 $(t_1 \geq t_0 > 0)$ in the formulation makes a difference between the L–S and G–L characterizations of the wave. The set $B(t)$ from Section 2 is also a support of the thermomechanical load Eq. (2). However, a domain of influence of the thermomechanical load at instant t is defined by Eq. (4), in which C satisfies the inequality: $C \geq \max(C_1', C_2')$, where $C_1' = C_1$ and

$C_2{}'$ is an upper bound over B of a simple algebraic function that depends on θ_0, t_0, t_1/t_0, ρ, c_E, $|\mathbf{M}|$ and $|\mathbf{K}|$ (cf. Theorem 1).

With regard to a mixed displacement-temperature problem (MDTP) of G–L theory, the following theorem holds true:

Theorem 2 (Domain of influence theorem for MDTP of G–L theory)**.** Let ($\mathbf{u}$, θ) be a solution to MDTP of G–L theory. Then

$$\mathbf{u} = 0,\ \theta = 0 \text{ on } [\bar{B} - B^*(t)] \times [0,\ t], \tag{7}$$

where $B^*(t)$ is given by Eq. (4), in which C satisfies the inequality $C \geq \max(C_1{}',\ C_2{}')$

Proof ((Carbonaro and Ignaczak, 1987))**.** A physical interpretation of Theorem 2 is similar to that of Theorem 1. Moreover, the definition of C implies that the velocities $C_1{}'$ and $C_2{}'$ correspond, respectively, to the maximum speed of a quasimechanical and of a quasithermal wave propagating in the G–L model; and, for $|\mathbf{M}| = 0$, they reduce to the maximum speeds of a pure mechanical and a pure thermal wave, respectively.

Also, for a particular nonhomogeneous thermoelastic G–L model in which the acoustic and heat conductivity tensor fields are relatively small, the maximum speed of a thermoelastic wave is given by

$$C_0{}' = \sup_B \left\{ \frac{t_1}{t_0} \left(\frac{\theta_0}{\rho c_E} \right)^{\frac{1}{2}} |\mathbf{M}| \right\}. \tag{8}$$

In addition, if (C_1, C_2) stands for a pair of velocities in the L–S theory, and if the thermomechanical constitutive fields of the L–S and G–L models are identical, by virtue of the inequality: $t_1 \geq t_0 > 0$, we obtain $C_2{}' \geq C_2$, and $C_2{}' = C_2$ if and only if $t_1 = t_0 > 0$. Therefore, the following observations are in order. If the supports of the thermomechanical load in MDTP of the L–S and G–L theories are the same for a fixed time t, and the constitutive fields in both theories are the same, then:

(i) the domain of influence of G–L theory is not smaller than that of L–S theory, and
(ii) the domain of influence of G–L theory coincides with that of L–S theory if $t_1 = t_0 > 0$.

Clearly, Theorem 2 covers a conventional MDTP of G–L theory in which the initial conditions are imposed on the pair ($\mathbf{u}$, θ). For a domain of influence theorem associated with a pure stress-temperature initial-boundary value problem of G–L theory that admits initially distributed thermoelastic defects, the reader is referred to Ignaczak (1991).

Finally, we note that a particular domain of influence results in the L–S and G–L theories have been obtained in a number of papers devoted to the potential-temperature waves propagating in a homogeneous isotropic thermoelastic body. This type of waves is generated by a pair ($\mathbf{u}$, θ) in which $\mathbf{u} = \nabla\Phi$ and Φ is a scalar potential on $\bar{B} \times [0, \infty)$ and, for a one-dimensional domain B, it covers all transient plane thermoelastic waves in the L–S and G–L theories. In a one-dimensional initial-boundary value problem for a semi-space $x \geq 0$ subject to a thermomechanical load on the boundary $x = 0$, a domain of influence is identified with the boundary layer $0 \leq x \leq v_2 t$, where v_2 is the greater speed of a decomposition theorem of the G–L theory (Ignaczak, 1991).

A study of one-dimensional thermoelastic waves produced by an instantaneous plane source of heat in homogeneous isotropic infinite and semi-infinite models of G–L type is presented by Hetnarski and

Ignaczak (1993), while the application of this study to the analysis of the response of a semi-infinite G–L model to short laser pulses is given by Hetnarski and Ignaczak (1994).

The mathematical theory of a potential-temperature initial-boundary value problem that accommodates asymptotic behavior of the waves analyzed by Hetnarski and Ignaczak (1993, 1994) as $t \to \infty$ was presented by the late Prof. Gaetano Fichera in two papers (Fichera, 1997a, 1997b); while a comparison between the wave-forms observed experimentally in an aluminum plate subject to a high-power Nd–YAG laser pulse and the thermoelastic waves propagating in a G–L plate loaded by a laser induced heat supply, is given by Suh and Burger (1998a) (see also Suh and Burger, 1998b).

4. A thermoelastic wave propagating in the H–I model

The H–I model proposed by Hetnarski and Ignaczak has been introduced in an attempt to describe low-temperature soliton-like thermoelastic waves (Ignaczak, 1990; Hetnarski and Ignaczak, 1996; Hetnarski and Ignaczak, 1997). The model is characterized by a system of nonlinear field equations in which, in comparison to the system of classical coupled thermoelasticity, both the free energy and the heat flux vector depend not only on the absolute temperature and the strain tensor but also on 'elastic' heat flow that satisfies an evolution equation, and enters a modified Fourier law and a modified free energy formula through a linear term and a quadratic term, respectively.

For a three-dimensional isotropic homogeneous body, a thermoelastic wave is described by a triplet $(T, \mathbf{U}, \mathbf{B})$ on $\bar{B} \times [0, \infty)$ that satisfies a nonlinear coupled system of field equations subject to linear initial and boundary conditions. In this description, T, $\mathbf{U}$ and $\mathbf{B}$ denote the absolute temperature, displacement vector and elastic heat flow vector fields, respectively. The physical properties of the model are represented by the set of parameters: $\{\omega, \varepsilon^*, \zeta, \kappa\}$, in which ω, ε^*, ζ and κ stand for a low-temperature parameter ($\omega \ll 1$), a generalized thermoelastic coupling constant, an inertia coefficient and a function of the Poisson's ratio, respectively.

In a one-dimensional case in which: (i) the body occupies an infinite space, i.e. $|x| \leq \infty$, (ii) the body force and heat supply fields are absent, (iii) the triplet $(T, \mathbf{U}, \mathbf{B})$ is given by: $T = T(x,t)$, $\mathbf{U} = [u(x, t), 0, 0]$ and $\mathbf{B} = [\Phi_x, 0, 0]$, where $\Phi = \Phi(x,t)$ is an elastic heat flow potential, the three-dimensional field equations reduce to a nonlinear system of field equations for a pair (u, Φ) involving the parameters ω, ε^* and ζ only. The temperature $T = T(x, t)$ the total heat flux in the x-direction $q = q(x, t)$ and the stress $\Sigma = \Sigma(x, t)$ in the x-direction are generated by (u, Φ) in a unique way.

The nonlinear system of equations for (u, Φ) involves the small parameter ω, so an asymptotic analysis may be used to obtain approximate solutions to these equations. The case $\omega = 0$ corresponds to a thermodynamical equilibrium at which $T = 1$, $q = 0$ and $\Sigma = 0$. Also, when $u = 0$ and $\epsilon^* = 0$, the one-dimensional equations reduce to those describing a low-temperature nonlinear rigid heat conductor (cf., Hetnarski and Ignaczak, 1995).

Let $s = x - vt$, where v is a positive constant. A soliton-like thermoelastic wave is defined as a triplet $[T(s), q(s), \Sigma(s)]$ generated by a pair $[u(s), \Phi(s)]$ that satisfies the one-dimensional field equations for $|s| \leq \infty$, subject to the boundary conditions: $T(-\infty) = T(+\infty) = 1$, $q(-\infty) = q(+\infty) = 0$ and $\Sigma(-\infty) = \Sigma(+\infty) = 0$. Clearly, a soliton-like thermoelastic wave is a localized wave propagating with a constant velocity v in the x-direction. The following theorem holds true:

Theorem 3.

(i) If $\hat{\zeta} \equiv \zeta\omega^{-1/2} \geq 1$ and $\epsilon^* > 0$, then there are two implicit-form soliton-like thermoelastic waves $[T(s_1), q(s_1), \Sigma(s_1)]$ and $[T(s_2), q(s_2), \Sigma(s_2)]$ propagating with velocities v_1 and v_2, respectively, in the x-direction ($s_i = x - v_i t$, $i = 1, 2$); and v_1 and v_2 are represented by simple algebraic functions of ω, ε^* and $\hat{\zeta}$.

(ii) If ω and ζ are both independent of each other and relatively small, then there are two closed-form fast-moving soliton-like thermoelastic waves $[T^*(s_1), q^*(s_1), \Sigma^*(s_1)]$ and $[T^*(s_2), q^*(s_2), \Sigma^*(s_2)]$, each revealing a fountain effect in a neighborhood of the moving front, and each close to a thermodynamical equilibrium far from the front; two self-equilibrated forces parallel to the direction of motion and applied to the wall in a neighborhood of the moving front $s_i = s*_i = \text{const}$, secure thermodynamical equilibrium of the wave $[T^*(s_i), q^*(s_i), \Sigma^*(s_i)]$.

Proof ((Hetnarski and Ignaczak, 1995, 1996, 1997)**.** The soliton-like thermoelastic waves $[T(s_1), q(s_1), \Sigma(s_1)]$ and $[T(s_2), q(s_2), \Sigma(s_2)]$ that occur in Theorem 3 represent the quasi-thermal and quasi-mechanical waves, respectively (Hetnarski and Ignaczak, 1997). Therefore, Theorem 3 may be useful in a low-temperature thermoelastic solid experiment in which both types of such waves are observed.

It should be noted that the model of a low-temperature nonlinear thermoelastic solid discussed here is confined to a homogeneous isotropic body with material properties independent of temperature. This is a severe restriction on the results following from the discussion of the model since in reality at low temperatures these properties depend strongly on temperature (cf., Zemansky, 1964). The aim of this section however, was to discuss a simple model of a low-temperature nonlinear thermoelastic solid for which soliton-like closed-form solutions may be obtained. A low-temperature thermoelastic solid, similar to that considered in this section but with temperature-dependent properties, may be treated by numerical methods.

As far as other low-temperature thermoelastic models proposed in the literature are concerned, the papers by Caviglia et al. (1992) and by Kosinski et al. (1997) should be mentioned.

5. A thermoelastic wave propagating in G–N model

The G–N model, proposed by Green and Naghdi (1993) is described by a system of PDE in which, in comparison to the classical thermoelastic system, the Fourier law of heat conduction is replaced by a heat flux rate-temperature gradient relation. So, a thermoelastic wave propagating in G–N model, and corresponding to a displacement-temperature initial-boundary value problem, is characterized in terms of a pair $(\mathbf{u}, \theta)$ that satisfies the displacement-temperature field equations in which the energy equation does not contain the temperature rate $\dot{\theta}$. As a result, a solution $(\mathbf{u}, \theta)$ to the problem represents an undamped thermoelastic wave, and this motivates the name of G–N theory as Thermoelasticity Without Energy Dissipation (TWED). For the L–S model the displacement-temperature energy equation does contain $\dot{\theta}$; similarly, $\dot{\theta}$ is included in the displacement-temperature energy equation of the G–L theory. This is a reason why the L–S and G–L models represent materials transmitting damped thermoelastic waves. The existence of damped thermoelastic waves in the L–S and G–L theories has been revealed in a number of papers devoted to theoretical (Achenbach, 1968; Ignaczak, 1978) and applied aspects (Ignaczak, 1989; Hetnarski and Ignaczak, 1993; Hetnarski and Ignaczak, 1994) of these theories.

A Saint-Venant's principle associated with an initial-boundary value problem of the G–N theory for a homogeneous isotropic thermoelastic body was presented by Nappa (1998). The principle asserts that a measure of thermoelastic energy $E = E(r, t)$ for the problem has the properties:

(i) $E(r, t) = 0$ for $r \geq ct$, and
(ii) $E(r, t) \leq E(0, t)(1 - r/ct)$ for $r \leq ct$;

where r is the distance of a point of the body B from the thermomechanical load support $B(t)$ (see Section 2), and c is a constant of the velocity dimension. Clearly, the property **(i)** is a form of the domain of influence result for the problem, and **(ii)** represents a spatial decay estimate of the energy $E(r, t)$ for $r \leq ct$, the decay rate being controlled by the factor $(1-r/ct)$. The Nappa result is similar to that of classical isothermal elastodynamics (Chirita and Quintanilla, 1996). This analogy could be expected, as both the isothermal elastic waves and the thermoelastic waves of G–N theory propagate without energy dissipation.

A uniqueness theorem for a natural stress-entropy flux initial-boundary value problem of the G–N theory was proved by Chandrasekharaiah (1996a), while the continuous dependence of a solution to the displacement-temperature initial-boundary value problem on the thermomechanical load in this theory was established by Iesan (1998). The undamped character of one-dimensional thermoelastic waves in the G–N theory was discussed by Chandrasekharaiah (1996b).

6. A thermoelastic wave propagating in the C–T model

The C–T model proposed by Chandrasekharaiah and Tzou in 1998 (Tzou, 1995; Chandrasekharaiah, 1998) is such an extension of the classical thermoelastic model in which the Fourier law is replaced by an approximation to a modified Fourier law with two different time translations: a phase-lag of the heat flux τ_q and a phase-lag of the temperature gradient τ_θ. A Taylor series approximation of the modified Fourier law, together with the remaining field equations, leads to a complete system of equations describing a dual-phase-lag thermoelastic model. The model transmits thermoelastic disturbances in a wave-like manner if the approximation is linear with respect to τ_q and τ_θ, and $0 \leq \tau_\theta < \tau_q$; or quadratic in τ_q and linear in τ_θ, with $\tau_q > 0$ and $\tau_\theta > 0$. In the former case, the linear approximation of the modified Fourier law together with the energy balance equation for a rigid heat conductor lead to Jeffreys type hyperbolic heat conduction equations (Joseph and Preziosi, 1989, 1990; Tamma and Zhou, 1998). Also, in this case the following decomposition theorem for the heat flux holds true.

Theorem 4. If the second order tensor $\mathbf{K}$ in the linear approximation of the modified Fourier law takes the form: $\mathbf{K} = (\tau_q/\tau_\theta)\mathbf{K}_\mathrm{F}$, $0 < \tau_\theta < \tau_q$, where $\mathbf{K}_\mathrm{F}$ is the heat conductivity tensor of the classical Fourier model in which the heat flux $\mathbf{q}_\mathrm{F}$ is given by $\mathbf{q}_\mathrm{F} = -\mathbf{K}_\mathrm{F}\nabla\theta$; and if $\mathbf{K}_\mathrm{C} = -(1-\tau_q/\tau_\theta)\mathbf{K}_\mathrm{F}$ stands for the heat conductivity tensor of L–S model, in which the heat flux $\mathbf{q}_\mathrm{C}$ satisfies the Cattaneo law: $\mathbf{q}_\mathrm{C} + \tau_q\dot{\mathbf{q}}_\mathrm{C} = -\mathbf{K}_\mathrm{C}\nabla\theta$, then the heat flux $\mathbf{q}$ admits the representation: $\mathbf{q} = \mathbf{q}_\mathrm{F} + \mathbf{q}_\mathrm{C}$ and $\mathbf{K} = \mathbf{K}_\mathrm{F} + \mathbf{K}_\mathrm{C}$.

Proof. is obtained by substituting $\mathbf{q}$ in the form $\mathbf{q} = \mathbf{q}_\mathrm{F} + \mathbf{q}_\mathrm{C}$ into the approximation of the modified Fourier law and using the definitions of the pairs $(\mathbf{q}_\mathrm{F}, \mathbf{q}_\mathrm{C})$ and $(\mathbf{K}_\mathrm{F}, \mathbf{K}_\mathrm{C})$.

For an isotropic rigid heat conductor, Theorem 4 was proved by Tamma and Zhou (1998). In the Tamma–Zhou terminology, τ_q and τ_θ are called the relaxation and retardation times, respectively.

For a C–T model based on the Taylor series approximation of the modified Fourier law which is quadratic in τ_q and linear in τ_θ, a displacement-temperature initial-boundary value problem involving the third order time derivatives of unknown solution may be formulated. Such a problem is a natural extension of the displacement-temperature problem of L–S theory (see Section 2).

Similarly, as in the case of the L–S and G–L models, no combined experimental data are available that might be used to determine the time parameters τ_q and τ_θ as well as the remaining thermomechanical properties of a C–T model.

Finally, we note that only particular one-dimensional initial-boundary value problems have been solved in the C–T theory (Chandrasekharaiah, 1998). A general domain of influence theorem, as well as a principle of Saint-Venant's type for the dual-phase-lag thermoelastic model, have not been obtained up to date.

7. Concluding remarks

1. Five different models of a thermoelastic solid in which disturbances are transmitted in a wave-like manner have been reviewed. These are the L–S, G–L, H–I, G–N, and C–T models. Except for the H–I model which is strongly nonlinear and applicable at low temperatures, the remaining models are linear.
2. The emphasis has been made on upper bounds for the velocities of waves propagating in a nonhomogeneous anisotropic solid in which thermoelastic coupling cannot be ignored. To this category of solids belong the L–S and G–L models in which the maximum speed of a thermoelastic wave is dominated by a suitably scaled stress-temperature tensor field, provided the acoustic and heat conductivity tensor fields are relatively small.
3. Two fast moving soliton-like thermoelastic waves, each revealing a fountain effect in a neighborhood of a moving front, and each close to thermodynamical equilibrium far from the front have been exposed for the low-temperature nonlinear H–I model. The model could be a starting point for working out a new model that transmits soliton thermoelastic waves of classical type.
4. The G–N model as opposed to the L–S and G–L models admits only propagation of undamped thermoelastic waves, and this motivates the name of the G–N theory as a Thermoelasticity Without Energy Dissipation. For the G–N model a Saint-Venant's principle analogous to that of an isothermal linear elastodynamics holds true.
5. The C–T model is an extension of the L–S model in the sense that both a phase-lag of the heat flux τ_q and a phase-lag of the temperature gradient τ_θ come into the formulation of an initial-boundary value problem; and depending on the Taylor series approximation of a modified Fourier law of heat conduction, one obtains a sequence of approximate initial-boundary value problems of the C–T theory.

The results presented in this survey should prove useful for researchers in material science, designers of new materials, low-temperature physicists, as well as for those working on the development of a theory of hyperbolic thermoelasticity.

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